

Dynamics of scalar fields in the background of rotating black holes II: A note on superradiance

Nils Andersson^(1,2), Pablo Laguna⁽³⁾ and Philippos Papadopoulos⁽⁴⁾

⁽¹⁾ Institut für Astronomie und Astrophysik

Universität Tübingen, D-72076 Tübingen, Germany

⁽²⁾ Department of Mathematics, University of Southampton, Southampton, UK

⁽³⁾ Department of Astronomy & Astrophysics and

Center for Gravitational Physics & Geometry

Penn State University, University Park, PA 16802, USA

⁽⁴⁾ Max-Planck-Institut für Gravitationsphysik

Schlaatzweg 1, 14473 Potsdam, Germany

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We analyze the amplification due to so-called superradiance from the scattering of pulses off rotating black holes as a numerical time evolution problem. We consider the “worst possible case” of scalar field pulses for which superradiance effects yield amplifications $< 1\%$. We show that this small effect can be isolated by numerically evolving quasi-monochromatic, modulated pulses with a recently developed Teukolsky code. The results show that it is possible to study superradiance in the time domain, but only if the initial data is carefully tuned. This illustrates the intrinsic difficulties of detecting superradiance in more general evolution scenarios.

I. INTRODUCTION

In the last few years, we have been involved in the development of a numerical code for the time evolution of perturbations of rotating black holes based on the Teukolsky equation [1,2]. There are several motivations for this work. One is the desire to revisit problems that have previously (mainly in the 1970s) only been approached in the frequency domain. That is, our goal is to explore the effects of the rotation of the black hole from a “time-evolution” point of view. More importantly, our ultimate goal is to provide a framework that will, once we understand how to construct astrophysically relevant initial data [3], be used to extend the close-limit approximation of head-on black hole collisions to the case of inspiral black hole mergers. The working premise in head-on, close-limit collisions [4] is that the merger can be viewed as perturbations of non-rotating black holes. In contrast, an inspiral close-limit approximation requires perturbations about a rotating black hole.

Our Teukolsky code project took us first to study the dynamics of scalar fields in the Kerr geometry [1]. This work mainly concerned the late-time, power-law behaviour of a scalar perturbation. The second installment concerned gravitational perturbations [2] and discussed the full dynamical response of a black hole to an external perturbation, namely the quasinormal mode ringing and the subsequent late-time tails. In Ref. [2], we also dealt briefly with superradiance: The anticipated amplification as certain wavelengths are scattered by the black hole. However, although the results we obtained indicated the presence of superradiance [2], we feel that our previous analysis was not completely satisfactory. Hence, the goal of this short paper is to return to the issue of superradiance in a setting that yields unequivocal evidence for the superradiance phenomenon.

The direct approach to measure superradiance from the time evolution of perturbations of rotating black holes is to compute the energy flux going “down the hole”. For perturbative fields that posses well-defined stress-energy tensors (e.g. scalar and electromagnetic fields), it is possible to construct such a conserved energy flux [5]. The case of gravitational perturbations is not that simple [5,6]. For this reason, we will concentrate our analysis on the “simple” case of scalar perturbations. The price to pay is that superradiant effects in this case are $< 1\%$ [7], thus requiring a highly accurate evolution code.

For scalar perturbations, the Teukolsky equation in Boyer-Lindquist coordinates reads

$$\begin{aligned} & \left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] \frac{\partial^2 \Phi}{\partial t^2} + \frac{4iMamr}{\Delta} \frac{\partial \Phi}{\partial t} - \frac{\partial}{\partial r} \left(\Delta \frac{\partial \Phi}{\partial r} \right) \\ & - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) - m^2 \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right] \Phi = 0 . \end{aligned} \quad (1)$$

Above, M is the mass of the black hole, a is its angular momentum per unit mass and $\Delta \equiv r^2 - 2Mr + a^2$. The two horizons of the black hole follow from $\Delta = 0$, and correspond to $r_{\pm} = M \pm \sqrt{M^2 - a^2}$. Reference to the azimuthal angle φ has been removed by assuming that the perturbation has a harmonic dependence $e^{im\varphi}$.

II. SUPERRADIANCE IN THE FREQUENCY DOMAIN

In the standard approach to solve the Teukolsky equation, one proceeds via separation of variables. For our present purposes, it is sufficient to note that this essentially corresponds to assuming that i) the time-dependence of the perturbation is accounted for via Fourier transformation, and ii) there exists a suitable set of angular function that can be used to separate the coordinates r and θ . In the case of scalar perturbations, the angular functions turn out to be standard spheroidal wave-functions [8]. Knowing this, we assume a representation (for each given integer m)

$$\Phi = \int d\omega e^{-i\omega t} \sum_{l=0}^{\infty} R_{lm}(r, \omega) S_{lm}(\theta, \omega) , \quad (2)$$

where it should be noted that the angular functions depend explicitly on the frequency ω ; that is, they are intrinsically time-dependent functions. After a separation of variables of the form given by (2), the problem reduces to a single ordinary differential equation for $R_{lm}(r, \omega)$. This equation can be written as

$$\frac{d^2 R_{lm}}{dr_*^2} + \left[\frac{K^2 + (2am\omega - a^2\omega^2 - E)\Delta}{(r^2 + a^2)^2} - \frac{dG}{dr_*} - G^2 \right] R_{lm} = 0 , \quad (3)$$

where $K = (r^2 + a^2)\omega - am$, $G = r\Delta/(r^2 + a^2)^2$, and the tortoise coordinate r_* is defined from $dr_* = (r^2 + a^2)/\Delta dr$. The variable E is the angular separation constant. In the limiting case $a \rightarrow 0$, it reduces to $l(l+1)$, and for nonzero a it can be obtained from a power series in $a\omega$ [9]. It should be noted that E is real valued for real frequencies.

The physical solution to (3) is defined by the asymptotic behaviour

$$R_{lm} \sim \begin{cases} \mathcal{T} e^{-i(\omega-m\omega_+)r_*} & \text{as } r \rightarrow r_+ , \\ \mathcal{R} e^{i\omega r_*} + e^{-i\omega r_*} & \text{as } r \rightarrow +\infty . \end{cases} \quad (4)$$

where $\omega_+ \equiv a/2M r_+$ is the angular velocity of the event horizon. \mathcal{T} and \mathcal{R} denote the transmission and reflection coefficients, respectively. Given the prescribed asymptotic behaviour, together with that for the complex conjugate of R_{lm} and the fact that two linearly independent solutions to (3) must lead to a constant Wronskian, it is not difficult to show that

$$(1 - m\omega_+/\omega)\mathcal{T} = 1 - \mathcal{R} . \quad (5)$$

From this result, it is evident that superradiance ($\mathcal{R} > 1$) occurs if

$$\omega < m\omega_+ = \frac{ma}{2Mr_+} . \quad (6)$$

Alternatively, one can deduce that energy can be extracted from the black hole immediately from the boundary condition (4). If $\omega < m\omega_+$, the solution $\propto e^{-i(\omega-m\omega_+)r_*}$, which behaves as “ingoing” into the horizon according to a local observer, will in fact correspond to waves coming out of the hole according to an observer at infinity. That is, for superradiant frequencies, one would expect to find energy flowing out from the horizon, cf. [10]. Superradiance is the wave analogue of the standard Penrose process, and its existence implies that it would in principle be possible to mine a rotating black hole for some of its rotational energy. This may seem exciting, but it is very unlikely that this effect will play a relevant role in any reasonable astrophysical scenario. Nonetheless, it is an interesting effect that deserves a close theoretical investigation.

In Fig. 1, we show a sample of results for the reflection coefficient in the case when $l = m = 2$. These results were obtained by a straightforward integration of (3) and subsequent extraction of \mathcal{R} . The maximum amplification in this case is close to 0.2 %. Similar results were obtained by Teukolsky and Press more than 25 years ago [5,7]. They also considered electromagnetic waves and gravitational perturbations and found that the maximum amplification is 0.3% for scalar waves, 4.4% for electromagnetic waves and as large as 138% for gravitational waves.

Given the results in Fig. 1, it is worth pointing out that they agree with the standard conclusions regarding the apparent “size” of a rotating black hole as seen by different observers. It is well-known (see for example [11]) that the black hole will appear larger to a particle moving around it in a retrograde orbit than to a particle in a prograde orbit. This is illustrated by the fact that the unstable circular photon orbit (at $r = 3M$ in the non-rotating case) is located at $r = 4M$ for a retrograde photon, while it lies at $r = M$ for a prograde photon. The results in Fig. 1 illustrate the same effect: In our case, we have prograde motion when ω/m is positive and retrograde motion when ω/m is negative. The data in Fig. 1 correspond to $m = 2$, and the enhanced reflection for positive frequencies as $a \rightarrow M$ has the effect that the black hole “looks smaller” to such waves. Conversely, the slightly decreased reflection for negative frequencies leads to the black hole appearing “larger” as $a \rightarrow M$.

III. SUPERRADIANCE IN THE TIME DOMAIN

From the discussion in the previous section, it should be clear what must be done to construct initial data sets that would yield superradiance with our Teukolsky code. Following an idea introduced in Ref. [2], we construct superradiant initial data by setting up a pulse containing mainly frequencies in the interval $0 < \omega < m\omega_+$ (from now on we will assume that m is positive). Obviously a non-superradiant pulse would be one whose main frequency content is outside this window. Furthermore, it makes the analysis easier and better suited to comparisons with frequency domain calculations if the pulse is “almost monochromatic.” To achieve this, we use as initial data an ingoing Gaussian pulse modulated by a monochromatic wave. Assuming that this initial pulse is centered far away from the black hole at $r_* = r_o$ and that the modulation frequency is σ , we set at $t = 0$

$$\Phi \propto e^{-(r_* - r_o + t)^2/b^2 - i\sigma(r_* - r_o + t)} . \quad (7)$$

Thus, the corresponding power spectrum is $P(\omega) = P_{\max} e^{-(\omega - \sigma)^2 b^2/4}$. The modulated pulse leads to a Gaussian frequency distribution that peaks at a frequency $\omega = \sigma$. Given this initial data, and the fact that there is no frequency dispersion in the Teukolsky equation, we ought to be able to detect superradiance in our evolutions if $0 < \sigma < m\omega_+$. Conversely, if $\sigma > m\omega_+$ or $\sigma < 0$ we should not find any amplification in the scattered waves. Finally, it is not enough for the peak of the power spectrum of the pulse to lie within the superradiant frequency window. To maximize the effect, we need also to minimize the “frequency overlap” of the initial pulse into the non-superradiant regime. To accomplish this, we have at our disposal the parameter b that governs the width of the pulse. If for instance we want $P(m\omega_+)/P_{\max} = \epsilon$, we should use

$$b = \frac{2\sqrt{\ln(1/\epsilon)}}{m\omega_+ - \sigma} . \quad (8)$$

Before showing that the pulse (7) can indeed be used to probe superradiance in the time domain, a few comments on our Teukolsky code are needed. The code was described in detail in Ref. [1], but there is one specific issue that is important for the present study that has not yet been discussed. To avoid numerical difficulties (cf. the description in [1]), we replace the azimuthal angle φ with the “ingoing Kerr-coordinate”, which is defined by

$$\tilde{\varphi} = \varphi + \int \frac{a}{\Delta} dr . \quad (9)$$

The transformation between the solution Φ to the standard Teukolsky equation and the solution $\tilde{\Phi}$ from our numerical code in terms of $\tilde{\varphi}$ is

$$\Phi = \tilde{\Phi} \exp \left[im \int \frac{a}{\Delta} dr \right] . \quad (10)$$

This is more or less obvious, but it is important to notice a few things. First of all, the replacement $\varphi \rightarrow \tilde{\varphi}$ changes the symmetry of the equations. While the original Teukolsky equation (1) is symmetric under the change $(m, \varphi) \rightarrow (-m, -\varphi)$, the equation we evolve with our code for $\tilde{\varphi}$ is not. That is, while evolutions for the same Gaussian pulse (unmodulated!) should lead to the same emerging scalar waves for $\pm m$ in the original case, this will not happen when we use $\tilde{\varphi}$. To ensure that the anticipated symmetries are present in our results, we have constructed initial data in Boyer-Lindquist coordinates and then used the transformation (10) to get data in the $\tilde{\varphi}$ coordinate system. This is especially important in the case of modulated Gaussians since the idea that we could enhance superradiance in the scattered wave by centering the Gaussian at a non-zero frequency was based on an analysis in Boyer-Lindquist coordinates, and the result of the experiment will depend on a careful tuning of the initial pulse.

Armed with the above conclusions, we are prepared to discuss our numerical results. Because we are interested in unveiling the amplification due to superradiance, we will focus on the energy flux through various surfaces surrounding the black hole.

Given a spacetime with a time Killing vector t^a (like the Kerr geometry) and a perturbation with a well-defined stress-energy tensor T_{ab} , it is possible to define [5] a conserved energy flux vector $T^a{}_b t^b$. The flux of energy across a 3-dimensional time-like hypersurface with unit normal r^a is then given by

$$dE = T_{ab} t^a r^b dS , \quad (11)$$

where dS is the 3-surface element of the hypersurface. For a massless scalar field,

$$T_{ab} = \frac{1}{2}(\nabla_a \bar{\Phi} \nabla_b \Phi + \nabla_a \Phi \nabla_b \bar{\Phi}) - \frac{1}{2}g_{ab} \nabla_c \Phi \nabla^c \bar{\Phi}, \quad (12)$$

with over-bars denoting complex conjugation. For simplicity, we monitor the flux of energy through $r = \text{constant}$ surfaces in Boyer-Lindquist coordinates. This assumption together with $r^a r_a = 1$ yield $r^a = \pm(0, \sqrt{\Delta}/\rho, 0, 0)$, where (as before) $\Delta = r^2 - 2Mr + a^2$ and $\rho^2 = r^2 + a^2 \cos^2 \theta$. Furthermore, the time Killing vector in this case reads $t^a = (1, 0, 0, 0)$, and the surface element is given explicitly by

$$dS = \sqrt{-g^{(3)}} \sin \theta d\theta d\varphi dt = \sqrt{\Delta} \rho \sin \theta d\theta d\varphi dt. \quad (13)$$

By collecting the above results and noticing that $t^a r_a = 0$, it is not difficult to show that

$$dE = \pm \frac{1}{2}(\partial_r \bar{\Phi} \partial_t \Phi + \partial_r \Phi \partial_t \bar{\Phi}) \Delta \sin \theta d\theta d\varphi dt. \quad (14)$$

Finally, integration over φ yields

$$dE = \pm \pi (\partial_r \bar{\Phi} \partial_t \Phi + \partial_r \Phi \partial_t \bar{\Phi}) \Delta \sin \theta d\theta dt, \quad (15)$$

which can be rewritten in terms of the tortoise coordinate as

$$dE = \pm \pi (\partial_{r_*} \bar{\Phi} \partial_t \Phi + \partial_{r_*} \Phi \partial_t \bar{\Phi}) (r^2 + a^2) \sin \theta d\theta dt. \quad (16)$$

In our simulations, the energy flux is monitored through two surfaces located at $r_* = \pm 20M$. The outer surface is well away from the black hole while the inner one is reasonably close to the event horizon. The scattering of waves by the curved spacetime should be strongest, i.e. most of the interesting physics should have its origin, in the region included between these surfaces.

In Figs. 2 and 3, we show the results of the superradiance “experiment”. The displayed data are for two qualitatively different situations. Both datasets correspond to evolutions with $m = 2$ and a black hole rotation parameter $a = 0.99M$. In both cases the pulse was initially centered around $r_o = 125M$, and the angular distribution of the initial data was chosen to be the standard spherical harmonic $Y_2^2(\theta, \varphi)$. The first case (see Fig. 2) shows a situation where one would expect to see superradiance. We have chosen the modulation frequency of the impinging Gaussian (see the previous section) such that $\sigma = 0.8 m \omega_+$, and the width of the Gaussian corresponds to $\epsilon = 0.01$. That is, the pulse has its main support (in the frequency domain) in the superradiant regime. The second case (Fig. 3) corresponds to a Gaussian with the same width but now centered around $\sigma = -0.8 m \omega_+$.

As is obvious from Fig. 1, the scattering of the two pulses we have chosen should be quite different. And not surprisingly, we find that this is indeed the case. In the superradiant case shown in Fig. 2, all the initial energy is reflected by the black hole. Superradiance is distinguished in two ways. By monitoring the energy flowing across the surface at $r_* = 20M$, we see that the reflected energy is slightly amplified after scattering (cf. the upper panel of Fig. 2). In this specific case, the amplification corresponds to 0.14%. It should be compared to the maximum single frequency amplification of 0.187% for $a = 0.99M$, deduced from the data in Fig. 1 (and also the 0.3% found by Teukolsky and Press [5]). That we are seeing superradiance is also clear from the fact that energy flows out through the surface at $r_* = -20M$ (cf. the lower panel of Fig. 2). The total energy flowing out through the inner surface corresponds to a superradiant amplification of 0.11%, in reasonable agreement with the result deduced at the outer surface.

The non-superradiant results, obtained by modulating the Gaussian with a frequency of opposite sign to that used in the superradiant case, are in clear contrast to the superradiant ones. In Fig. 3, there is certainly no amplification of the reflected wave. In fact, as can be seen from the upper panel of Fig. 3, the infalling pulse is almost entirely swallowed by the black hole. That there would be very little reflection in this case could, of course, be anticipated by comparing our chosen Gaussian pulse to the data in Fig. 1.

IV. CONCLUDING REMARKS

We have designed a numerical experiment that clearly exhibits the presence of superradiance phenomenon when waves of a certain character are scattered by a rotating black hole. Superradiant effects have previously only been studied in the frequency domain, and one conclusion that can be drawn from the present work is that superradiance is perhaps best approached in that way. True, we have managed to extract the amplification due to superradiance in the “worst possible case” of scalar waves, when the superradiant amplification is expected to be considerably less

than 1%, but this was mainly due to having at our disposal a conserved flux. More than anything else this is direct evidence of the precision of our evolution code to solve the Teukolsky equation [1,2].

The initial data we used to isolate the tiny effect due to superradiance is required to be “almost monochromatic” and thus artificial. Our numerical experiment shows that superradiance can play a role in evolutions when the scattered pulse has support only in a restricted frequency domain. This is undoubtedly an interesting illustration, but what about superradiance in more general cases? It seems to us that the effect is easiest to isolate if one monitors different frequencies separately, i.e. works in the frequency domain. The main reason for this is that an amplification of a reflected signal with increasing a is not in itself an indication of superradiance. The results shown in Fig. 1 indicate that one would generally expect enhanced reflection of prograde moving waves as $a \rightarrow M$. This effect is likely to overwhelm the actual “amplification” of certain superradiant frequencies in an evolution of general initial data. This is certainly true for scalar waves, and since the maximum amplification of impinging electromagnetic waves is only a few percent [5] the conclusion should hold also in that case. However, the possibility that superradiance may play a distinctive role in an “astrophysical” evolution for gravitational waves cannot be ruled out. For gravitational waves, the amplitude of certain frequencies should be amplified by more than a factor of two [5]. A detailed study of that case could provide interesting results, but the present results for scalar field prompts us to proceed with caution. We have learned that the initial data requires careful tuning in order that superradiance be observed. For general initial data, absorption of the non-superradiant frequencies will typically make the amplification due to superradiance difficult to distinguish. Moreover, superradiance should not be confused with the competing effect that the “size” of the black hole changes with the rate of rotation. As we have seen, this effect will generally lead to a much enhanced reflection of prograde waves which may confuse an attempt to distinguish superradiance.

V. ACKNOWLEDGMENTS

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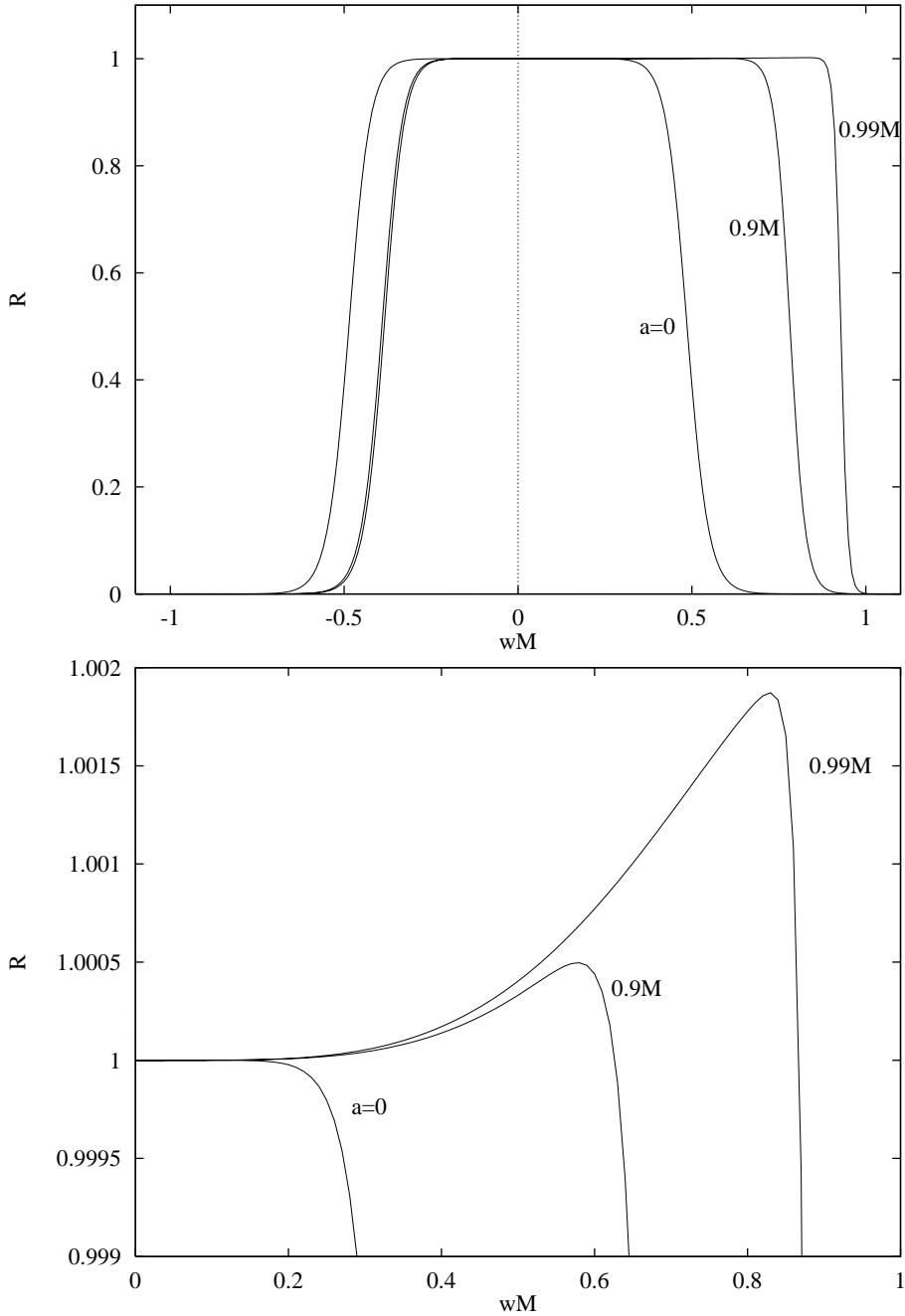


FIG. 1. Reflection coefficient (\mathcal{R}) as a function of the frequency of the wave (ωM) for different values of the angular momentum parameter (a) with $l = m = 2$. Superradiance is present in the interval $0 < \omega M < m a / 2 r_+$, with $r_+ = M + \sqrt{M^2 - a^2}$. The bottom panel is a close-up of this superradiance regime. The maximum observed superradiant amplification is $\sim 0.187\%$. As $a \rightarrow M$, there is a clearly enhanced reflection of prograde waves ($\omega > 0$) while the overall reflection of retrograde waves ($\omega < 0$) decreases somewhat.

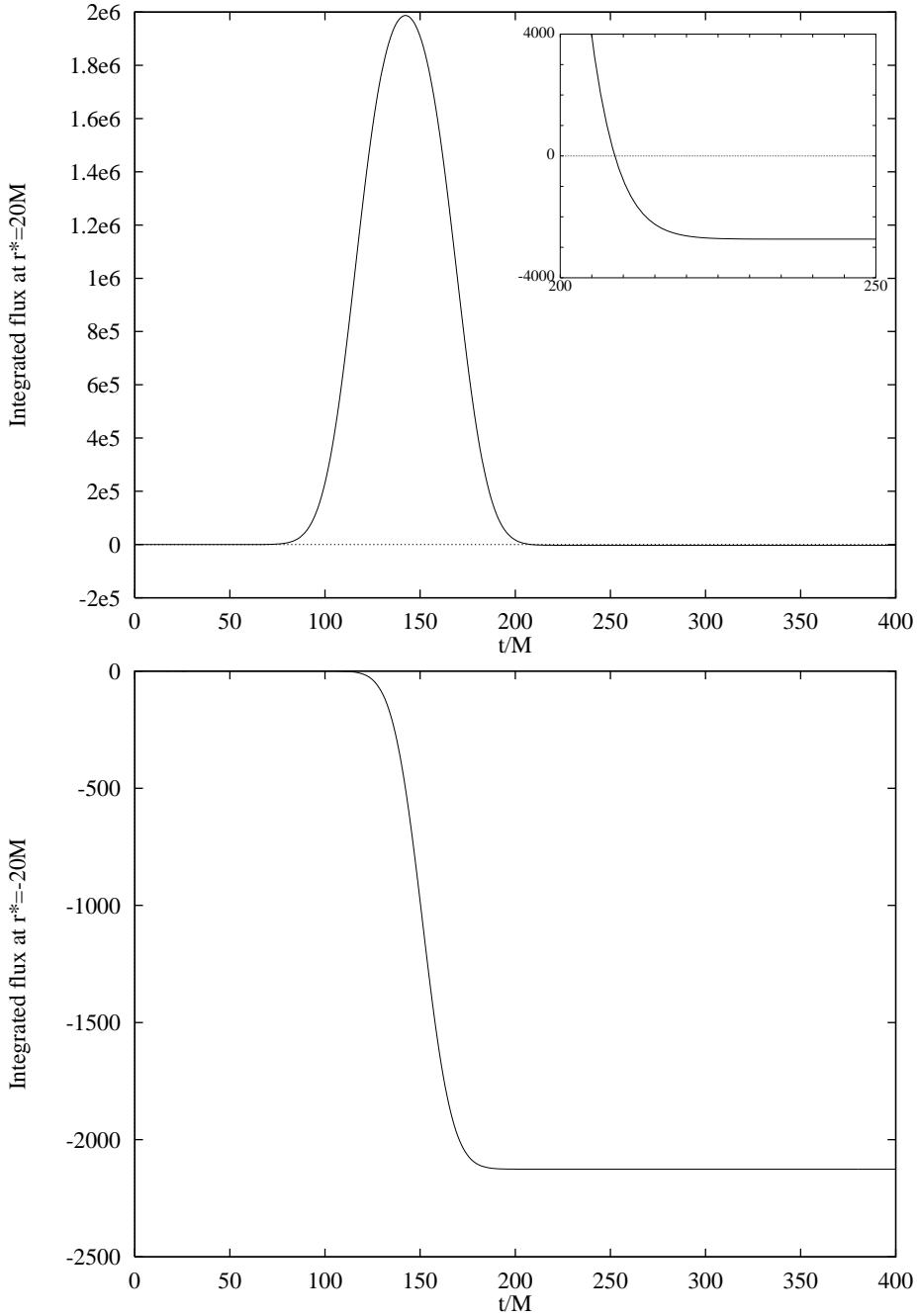


FIG. 2. An example of a superradiant evolution. We show results corresponding to a modulated Gaussian pulse (with $\sigma = 0.8 m \omega_+$, $m = 2$, $a = 0.99M$ and $\epsilon = 0.01$, see the main text for more details). We monitor the integrated energy flux through two surfaces, one located at $r_* = 20M$ and the second at $r_* = -20M$. (The normals of these surfaces are chosen such that energy flowing inwards into-the-hole across the outer surface is positive.) Superradiance manifests itself in two ways. At the outer surface (the upper panel), we see a total amplification of $\sim 0.14\%$ in the reflected energy. As is clear from the inset in the upper panel, the reflected energy is larger than what initially fell onto the black hole. At the inner surface (the lower panel), we find that energy mainly flows out of the horizon. The integrated flux of this energy corresponds to $\sim 0.11\%$ of the total energy falling onto the hole, i.e. agrees well with the amount that amplifies the reflected waves at the outer surface.

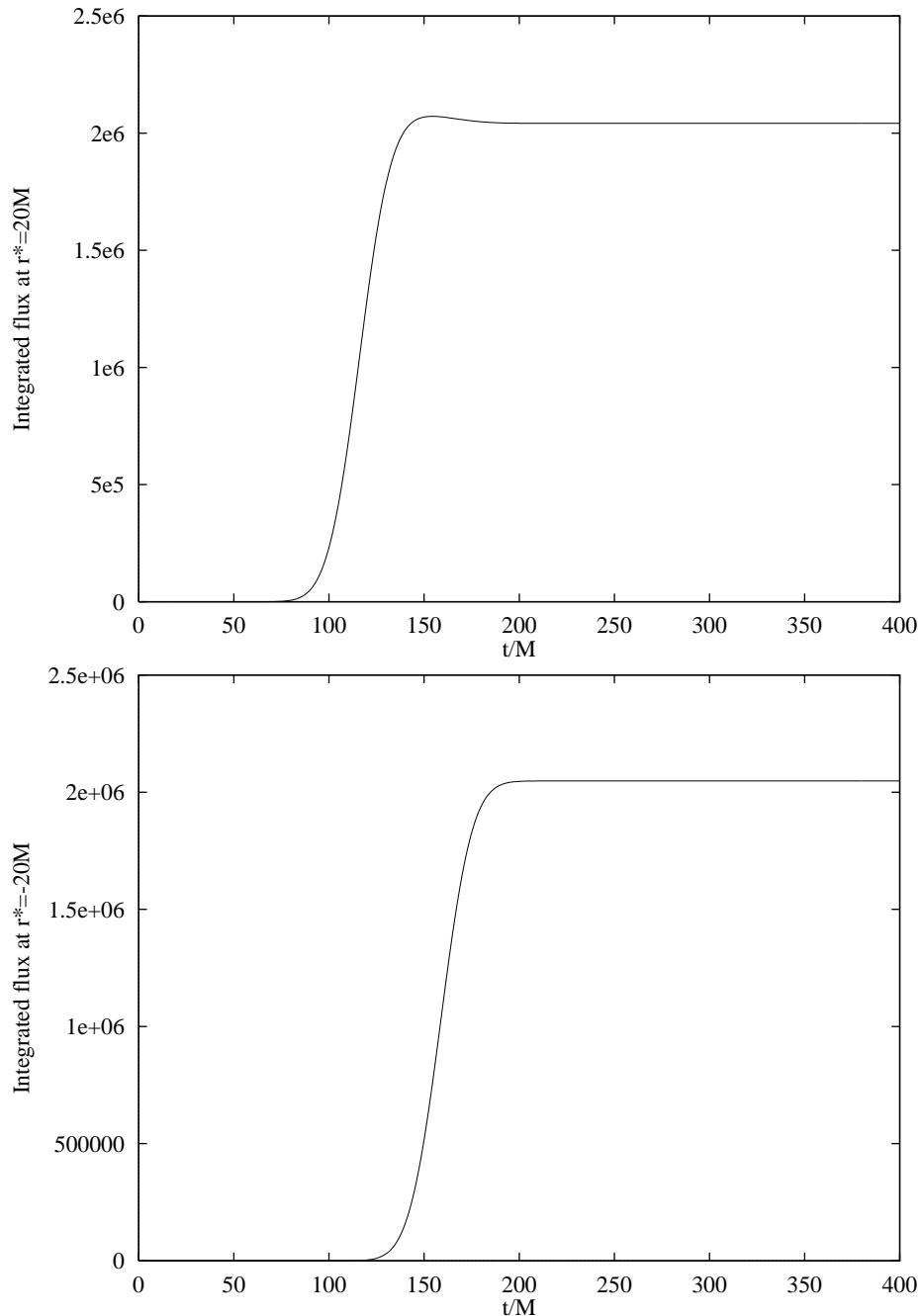


FIG. 3. An example of a non-superradiant evolution. The data are similar to that described in Fig. 2, but here the modulation frequency of the initial Gaussian is $\sigma = -0.8 m \omega_+$, and there is no sign of superradiance.